11007. Proposed by Western Maryland College Problems Group, Westminster, MY. Let ⟨⟩ denote Eulerian numbers, and let {} denote Stirling numbers of the second kind. Show that

\[ \sum_{j=1}^{n} 2^{j-1} \binom{n}{j} = \sum_{j=1}^{n} j! \left\{ \begin{array}{c} n \\ j \end{array} \right\}. \]  

(1)

Solution: Let \( \mathbb{N}_k \) denote the set of the first \( k \) positive integers. Then each term in the summation on the right-hand-side of (1) counts the number of onto functions \( f: \mathbb{N}_n \rightarrow \mathbb{N}_j \). This follows because the Stirling number of the second kind, \( \left\{ \begin{array}{c} n \\ j \end{array} \right\} \), counts the number of unordered partitions of a set of cardinality \( n \) into \( j \) non-empty classes. So \( j! \left\{ \begin{array}{c} n \\ j \end{array} \right\} \) gives the number of such ordered partitions, which is the same as the number of onto functions \( f: \mathbb{N}_n \rightarrow \mathbb{N}_j \).

In order to establish the identity, we need a connection between the number of onto functions and the Eulerian numbers. This is given in the following identity which appears in [1] using a different notation where its proof is left as an exercise.

Lemma 1.

\[ j! \left\{ \begin{array}{c} n \\ j \end{array} \right\} = \sum_{k=0}^{n-1} \binom{n}{k+1} \binom{k}{n-j}. \]  

(2)

Proof (of lemma). Since the left-hand side of (2) counts the number of onto functions from \( \mathbb{N}_n \rightarrow \mathbb{N}_j \), we show the right-hand side counts these functions as well.

So let \( f: \mathbb{N}_n \rightarrow \mathbb{N}_j \) be onto. Then \( f \) can be represented as an \( n \)-tuple of integers (\( a_1, a_2, \ldots, a_n \)) where \( 1 \leq a_i \leq j \) for each \( i \) and where \( \{ a_i | 1 \leq i \leq j \} = \mathbb{N}_j \) (since \( f \) is onto). Now rearrange the permutation so all the elements of \( f^{-1}(1) \) occur first and in increasing order, then the elements of \( f^{-1}(2) \) occur next arranged in increasing order, etc., to obtain \( (a_{i_1}, a_{i_2}, \ldots, a_{i_n}) \). Then \( (i_1, i_2, \ldots, i_n) \) is a permutation on \( \mathbb{N}_n \) with \( j-1 \) or fewer descents (where a descent occurs when one number in a permutation is less than its predecessor).

Now, given any \( n \)-permutation with \( j-1 \) or fewer descents, we use the descents as barriers to induce a partition on the \( n \) numbers. We add \( j-1-k \) additional barriers, where \( k \) is the number ofdescents, to construct an onto function \( f: \mathbb{N}_n \rightarrow \mathbb{N}_j \). This can be done by choosing where the \( j-1-k \) barriers go from
the $n - 1 - k$ available positions. This can be done in \( \binom{n-1-k}{j-1-k} = \binom{n-1-k}{n-j} \) ways. Summing over all possible $k$ gives the number of onto functions.

\[
\sum_{k=0}^{n-1} \binom{n}{k+1} \binom{n-1-k}{n-j} = \sum_{k=0}^{n-1} \binom{n}{k+1} \binom{k}{n-j}
\]

An example will serve to illustrate this counting method. A similar example is found in [2]. If we are counting the number of onto functions $f : \mathbb{N}_9 \to \mathbb{N}_6$ and consider the permutation on $\mathbb{N}_9$ given by 135274698, how many onto functions does this permutation correspond to with this counting method? If we write the permutation showing the descents, we get 135 ↓ 27 ↓ 469 ↓ 8. To define an onto function to $\mathbb{N}_6$, we must insert 2 additional barriers between adjacent numbers to create a total of 5 barriers to produce a partition into 6 ordered classes. Considering the available locations for additional barriers (denoted by $\sqcup$), 1 $\sqcup$ 3 $\sqcup$ 5 $\sqcup$ 2 $\sqcup$ 7 $\sqcup$ 4 $\sqcup$ 6 $\sqcup$ 9 $\sqcup$ 8, we must choose 2 of the 5 without regard to order in any of $\binom{5}{2}$ ways. \qed

Using the lemma, we have

\[
\sum_{j=1}^{n} j! \binom{n}{j} = \sum_{j=1}^{n} \sum_{k=0}^{n-1} \binom{n}{k+1} \binom{k}{n-j} = \sum_{j=1}^{n} \sum_{k=1}^{n} \binom{n}{k} \binom{k-1}{n-j} = \sum_{k=1}^{n} \binom{n}{k} \sum_{j=1}^{n} \binom{k-1}{n-j} = \sum_{k=1}^{n} \binom{n}{k} 2^{k-1}.
\]

References
